

SOME EXTENSIONS OF THE MEAN CURVATURE FLOW IN RIEMANNIAN MANIFOLDS

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ABSTRACT. Given a family of smooth immersions $F_t : M^n \rightarrow N^{n+1}$ of closed hypersurfaces in a locally symmetric Riemannian manifold N^{n+1} with bounded geometry, moving by the mean curvature flow, we show that at the first finite singular time of the mean curvature flow, certain subcritical quantities concerning the second fundamental form blow up. This result not only generalizes a recent result of Le-Sesum and Xu-Ye-Zhao, but also extends the latest work of N. Le in the Euclidean case (arXiv: math.DG/1002.4669v2).

1. INTRODUCTION

We study the blow-up phenomena of the geometric quantities for compact hypersurfaces M^n , $n \geq 2$, without boundary, which are smoothly immersed in a Riemannian manifold N^{n+1} . Let $M^n = M_0$ be given by some diffeomorphism

$$F_0 : U \subset \mathbb{R}^n \rightarrow F_0(U) \subset M_0 \subset N^{n+1}.$$

Consider a smooth one-parameter family of immersions

$$F(\cdot, t) : M^n \rightarrow N^{n+1}$$

satisfying the evolution equation

$$(1.1) \quad \frac{\partial}{\partial t} F(x, t) = -H(x, t)\nu(x, t), \quad \forall (x, t) \in M \times [0, T)$$

with the initial condition

$$F(\cdot, 0) = F_0(\cdot).$$

Here $H(x, t)$ and $\nu(x, t)$ denote the mean curvature and the unit outward normal vector of the hypersurface $M_t = F(M^n, t)$ at the point $F(x, t)$. Equation (1.1) is often called the mean curvature flow (MCF for short). We easily see that (1.1) is a quasilinear parabolic equation with a smooth solution at least on some short time interval.

When the ambient space is the Euclidean space \mathbb{R}^{n+1} , G. Huisken [8] showed that the norm of the second fundamental form $A(\cdot, t)$ of an evolving hypersurface under the mean curvature flow must blow up at a finite singular time.

Theorem 1.1 (G. Huisken [8]). *Suppose $T < \infty$ is the first singularity time for a compact mean curvature flow (1.1) with the ambient space $N^{n+1} = \mathbb{R}^{n+1}$. Then $\sup_{M_t} |A|(\cdot, t) \rightarrow \infty$ as $t \rightarrow T$.*

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Later, G. Huisken and C. Sinestrari [10] studied the blow up of H near a singularity for mean convex hypersurfaces. They also established lower bounds on the principal curvatures in this mean-convex setting.

After that, N. Le and N. Sesum [12], and H.-W. Xu, F. Ye and E.-T. Zhao [19] used different methods to extend the previous results. They independently showed that if the second fundamental form is uniformly bounded from below and the mean curvature is bounded in certain integral sense, then the mean curvature flow can be extended smoothly past time T . Their proofs are both based on blow-up arguments, and the Moser iteration using the Michael-Simon inequality [15].

Meanwhile N. Le and N. Sesum [12, 13], and H.-W. Xu, F. Ye and E.-T. Zhao [19] gave another global conditions for extending a smooth solution to the mean curvature flow. Specifically, they proved that

Theorem 1.2. *Suppose $T < \infty$ is the first singularity time for a mean curvature flow (1.1) of compact hypersurfaces of $N^{n+1} = \mathbb{R}^{n+1}$. Let p and q be positive numbers satisfying $\frac{n}{p} + \frac{2}{q} = 1$. If*

$$\|A\|_{L^{p,q}(M \times [0,T))} := \left[\int_0^T \left(\int_{M_t} |A|^p \right)^{q/p} dt \right]^{1/q} < \infty,$$

then this flow can be extended past time T . In particular, let $p = q = n + 2$, and if

$$\int_0^T \int_{M_t} |A|^{n+2} d\mu dt < \infty,$$

then this flow can be extended past time T .

Note that this result is still based on a blow-up argument, using the compactness theorem for the mean curvature flow [1]. Recently, N. Le [11] gave a logarithmic improvement of the above results by proving that a family of subcritical quantities concerning the second fundamental form blows up at the first finite singular time of the mean curvature flow.

Theorem 1.3 (N. Le [11]). *Suppose $T < \infty$ is the first singularity time for a mean curvature flow (1.1) of compact hypersurfaces of $N^{n+1} = \mathbb{R}^{n+1}$. If*

$$\int_0^T \int_{M_t} \frac{|A|^{n+2}}{\log(1 + |A|)} d\mu dt < \infty,$$

then this flow can be extended past time T .

If the ambient space N is a general Riemannian manifold, G. Huisken [9] gave an sufficient condition to assure the extension over time for mean curvature flow (1.1) provided the ambient space N admits bounded geometry. Here we say that a Riemannian manifold is called bounded geometry if its sectional curvature and the first covariant derivative of the curvature tensor are bounded, and if the injective radius is bounded from below by a positive constant. In [3], A.A. Cooper mainly investigated the behaviour at finite-time singularities of the mean curvature flow of compact Riemannian submanifolds $M^n \rightarrow N^{n+\alpha}$ ($\alpha \geq 1$) and generalized Huisken's results. He observed that in fact it suffices to consider the tensor $\bar{A}_{ij} = H^\alpha h_{ij\alpha}$, where $H = \text{tr}A$ is the mean curvature and h are the components of the second fundamental form. Namely, he proved that

Theorem 1.4 (A.A. Cooper [3]). *Let (N, h) be a Riemannian manifold with bounded geometry. Suppose $T < \infty$ is the first singular time for a mean curvature flow (1.1) of compact submanifolds of (N, h) . Then $\sup_{M_t} |\bar{A}|(\cdot, t) \rightarrow \infty$ as $t \rightarrow T$.*

In other words, from Theorem 1.4, we can see that the geometric quantity $\bar{A}(\cdot, t)$ uniformly bounded is enough to extend a mean curvature flow of compact Riemannian submanifolds $M^n \rightarrow N^{n+\alpha}$, provided N is a bounded geometry. Cooper's proof relies strongly on a blow-up argument combined with a compactness property of the mean curvature flow.

In the case of $\alpha = 1$, Cooper's result has been improved in [20], assuming certain integral bounds on the mean curvature and the second fundamental form is bounded from below.

Theorem 1.5 (H.-W. Xu, F. Ye and E.-T. Zhao [20]). *Let $F_t : M^n \rightarrow N^{n+1}$ ($n \geq 3$) be a solution to the compact mean curvature flow (1.1) of on a finite time interval $[0, T)$. If there is a positive constant C such that*

$$h_{ij} \geq -C$$

for $(x, t) \in M \times [0, T)$. Further we assume that

$$\|H\|_{L^\alpha(M \times [0, T))} := \left(\int_0^T \int_{M_t} |H|^\alpha d\mu dt \right)^{\frac{1}{\alpha}} < \infty$$

for some $\alpha \geq n + 2$. Then this flow can be extended past time T .

In this paper, we want to extend Theorems 1.2 and 1.3 to the case when the ambient space is a general Riemannian manifold. Similar to the logarithmic improvement result proved by N. Le [11], we obtain a family of subcritical quantities involving the second fundamental form blows up at the first finite singular time of the mean curvature flow. In particular,

Theorem 1.6. *Suppose $T < \infty$ is the first singularity time for a mean curvature flow (1.1) of compact hypersurfaces of a locally symmetric Riemannian manifold N^{n+1} with bounded geometry. If*

$$\int_0^T \int_{M_t} \frac{|A|^{n+2}}{\log(a + |A|)} d\mu dt < \infty,$$

where a is a fixed constant and $a \geq 1$, then this flow can be extended past time T .

Obviously, Theorem 1.6 includes Theorem 1.3 proved by N. Le in [11]. To prove Theorem 1.6, we mainly follow the ideas of the proof of Theorem 1.3 in [11], where a blow up argument and the Moser iteration are employed. In our setting, one difference of the proof of Theorem 1.3 is that the Michael-Simon inequality used in the Euclidean case should be replaced by the Hoffman-Spruck Sobolev inequality for submanifolds of a Riemannian manifold (see Lemma 3.1 below). Another difference is that curvature of the ambient space N^{n+1} will interfere with the evolution of the second fundamental form on M^n , and will may further interfere the Moser iteration process. However, if we assume the ambient space satisfies some special restrictions, then we can still go through along the Le's proof of Theorem 1.3.

Moreover, as a corollary Theorem 1.6 implies the following result.

Corollary 1.7. *Suppose $T < \infty$ is the first singularity time for a mean curvature flow (1.1) of compact hypersurfaces of a locally symmetric Riemannian manifold N^{n+1} with bounded geometry. If*

$$\int_0^T \int_{M_t} |A|^\alpha d\mu dt < \infty,$$

for some $\alpha \geq n + 2$, then this flow can be extended past time T .

Remark 1.8. Corollary 1.7 generalizes a recent result of H.-W. Xu, F. Ye and E.-T. Zhao (see Theorem 1.1 in [19]). When the ambient space N is the Euclidean space, we recover their result. Moreover Corollary 1.7 also extends a special case proved by N. Le and N. Sesum [13] (see Theorem 1.2 above).

Besides the above works, the closest precedent for the mean curvature flow is the Ricci flow introduced by R.S. Hamilton in his seminal paper [6] (see also [2]), where Hamilton showed that if $T < \infty$ is the maximal existence time of a closed Ricci flow solution $g(t)$, $t \in [0, T)$, then the supremum of the Riemannian curvature blows up as $t \rightarrow T$. In other words, a uniform bound for the Riemannian curvature on a finite time interval $M \times [0, T)$ is enough to extend Ricci flow past time T . Later, N. Sesum [17] employed Perelman's no local collapsing theorem [16] and improved the Hamilton's extension result. That is, if the norm of Ricci curvature is uniformly bounded over a finite time interval $[0, T)$, then we can extend the flow smoothly past time T . In [18], B. Wang improved the Sesum's result by a blow-up argument and the Moser iteration. He showed that if Ricci curvature is uniformly bounded from below and the scalar curvature is bounded in certain integral sense on a finite time interval $[0, T)$, then the Ricci flow can be extended past time T . Most recently, N. Le and N. Sesum in [14], and J. Enders, R. Müller and P. Topping in [5] independently showed that the Type I Ricci flow can be extended past time T if the scalar curvature is uniformly bounded over a finite time interval $[0, T)$.

The rest of this paper is organized as follows. In Sect. 2, we will give some basic notation and preliminary results. In Sect. 3, we will obtain Sobolev inequalities for the mean curvature flow in Riemannian manifolds. In Sect. 4, we will give the reverse Hölder and Harnack inequalities for a subsolution to a parabolic equation evolving by the mean curvature flow in Riemannian manifolds. In Sect. 5, we will prove Proposition 5.1, which means the second fundamental form can be bounded by its certain integral sense. In Sect. 6, we will apply Proposition 5.1 to finish the proof of Theorem 1.6. In Sect. 7, we will prove Corollary 1.7 using Theorem 1.6.

2. PRELIMINARIES

In the following sections, Latin indices range from 1 to n , Greek indices range from 0 to n and the summation convention is understood. Let

$$F(\cdot, t) : M^n \rightarrow N^{n+1}$$

be a one-parameter family of smooth hypersurfaces immersed in a Riemannian manifold N satisfying the evolution equation (1.1). We denote the induced metric and the second fundamental form on M by $g = \{g_{ij}\}$ and $A = \{h_{ij}\}$. The mean curvature of M is the trace of the second fundamental form, $H = g^{ij}h_{ij}$. The square of the second fundamental form is denoted by $|A| = g^{ij}g^{kl}h_{ik}h_{jl} = h_{ik}h^{ik}$. We write $\bar{R}m = \{\bar{R}_{\alpha\beta\gamma\delta}\}$ and $\bar{\nabla}\bar{R}m = \{\bar{\nabla}_\sigma\bar{R}_{\alpha\beta\gamma\delta}\}$ for the curvature tensor of N and its covariant derivative. Let ν be the outer unit normal to M_t . Then for a

fixed time t , we can choose a local field of frame e_0, e_1, \dots, e_n in N such that when restricted to M_t , we have $e_0 = \nu$, $e_i = \frac{\partial F}{\partial x_i}$. The relations between $A = \{h_{ij}\}$, Rm and $\bar{R}m$ are then given by the following equations of Gauss and Codazzi:

$$(2.1) \quad R_{ijkl} = \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk},$$

$$(2.2) \quad \nabla_k h_{ij} - \nabla_j h_{ik} = \bar{R}_{0ijk}.$$

In [9], we have the following evolution equations:

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= -2Hh_{ij}, \\ \frac{\partial}{\partial t} H &= \Delta H + H(|A|^2 + \bar{Ric}(\nu, \nu)), \\ \frac{\partial}{\partial t} |A|^2 &= \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + \bar{Ric}(\nu, \nu)) \\ &\quad - 4(h^{ij}h_j^m \bar{R}_{mli}{}^l - h^{ij}h^{lm} \bar{R}_{milj}) \\ &\quad - 2h^{ij}(\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l), \end{aligned}$$

where $\bar{Ric}(\nu, \nu) = \bar{R}_{0i0}{}^i$.

3. SOBOLEV INEQUALITIES FOR THE MCF

In this section we first introduce the Hoffman-Spruck Sobolev inequality. Then we apply it to prove the following important Lemma 3.3, which is essential in the proof of a Sobolev inequality, Proposition 3.5, for the mean curvature flow (1.1). More importantly, this Sobolev inequality will be crucial for the Moser iteration in the next sections.

Now we present the following Hoffman-Spruck Sobolev inequality for Riemannian submanifolds in [7].

Lemma 3.1 (D. Hoffman and J. Spruck [7]). *Let $M \rightarrow N$ be an isometric immersion of Riemannian manifolds of dimension n and $n+p$, ($p \geq 1$), respectively. Some notations are adopted as before. Assume $K_N \leq b^2$ and let h be a non-negative C^1 function on M vanishing on ∂M . Then*

$$(3.1) \quad \left(\int_M h^{n/(n-1)} dV_M \right)^{(n-1)/n} \leq c(n) \int_M [|\nabla h| + h|H|] dV_M,$$

provided

$$(3.2) \quad b^2(1-\alpha)^{-2/n} (\omega_n^{-1} Vol(\text{supp} h))^{2/n} \leq 1$$

and

$$(3.3) \quad 2\rho_0 \leq \bar{R}(M),$$

where

$$(3.4) \quad \rho_0 = \begin{cases} b^{-1} \sin^{-1} \left[b(1-\alpha)^{-1/n} (\omega_n^{-1} Vol(\text{supp} h))^{1/n} \right] & \text{for } b \text{ real,} \\ (1-\alpha)^{-1/n} (\omega_n^{-1} Vol(\text{supp} h))^{1/n} & \text{for } b \text{ imaginary.} \end{cases}$$

Here α is a free parameter, $0 < \alpha < 1$, and

$$(3.5) \quad c(n) := c(n, \alpha) = \pi \cdot 2^{n-1} \alpha^{-1} (1-\alpha)^{-1/n} \frac{n}{n-1} \omega_n^{-1/n}.$$

Remark 3.2. In Lemma 3.1, we may replace the assumption $h \in C^1(M)$ by $h \in W^{1,1}(M)$. As the mentioned remark in [7], the optimal choice of α to minimize c is $\alpha = n/(n+1)$. When b is real we may replace condition (3.3) by the stronger condition $\bar{R} \geq \pi b^{-1}$. When b is a pure imaginary number and the Riemannian manifold N is simply connected and complete, $\bar{R}(M) = +\infty$. Hence conditions (3.2) and (3.3) are automatically satisfied.

Following the proof of Lemma 2.1 in [11] by means of Lemma 3.1, we have the following general result.

Lemma 3.3. *Let M be a compact n -dimensional hypersurface without boundary, which is smoothly embedded in N^{n+1} . Assume $K_N \leq b^2$. Let*

$$(3.6) \quad Q = \begin{cases} \frac{n}{n-2} & \text{if } n > 2 \\ < \infty & \text{if } n = 2 \end{cases}$$

Then, for all non-negative Lipschitz functions v on M , we have

$$\|v\|_{L^{2Q}(M)}^2 \leq c_n \left(\|\nabla v\|_{L^2(M)}^2 + \|H\|_{L^{\frac{2(n+3)}{3}}(M)}^{\frac{2(n+3)}{3}} \|v\|_{L^2(M)}^2 \right)$$

provided the function $h := v^{\frac{2(n-1)}{n-2}}$ satisfies conditions (3.2) and (3.3), where H is the mean curvature of M and c_n is a positive constant depending only on n .

Remark 3.4. In Lemma 3.3, when b is a pure imaginary number and the Riemannian manifold N is simply connected and complete, $\bar{R}(M) = +\infty$. Hence conditions (3.2) and (3.3) are automatically satisfied.

Similar to the proof of Proposition 2.1 in [11], using Lemma 3.3 and Hölder's inequality, our Sobolev type inequality for the mean curvature flow in Riemannian manifolds is stated in the following proposition.

Proposition 3.5. *For all non-negative Lipschitz functions v , we have*

$$\begin{aligned} & \|v\|_{L^\beta(M \times [0, T])}^\beta \\ & \leq c_n \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^{4/n} \left(\|\nabla v\|_{L^2(M \times [0, T])}^2 + \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^2 \|H\|_{L^{\frac{2(n+3)}{3}}(M \times [0, T])}^{\frac{2(n+3)}{3}} \right) \end{aligned}$$

provided the function $h := v^{\frac{2(n-1)}{n-2}}$ satisfies conditions (3.2) and (3.3) for any $t \in [0, T]$, where $\beta := \frac{2(n+2)}{n}$.

4. REVERSE HÖLDER AND HARNACK INEQUALITIES

In this section we can follow the lines of [11] or [12], and easily obtain a soft version of reverse Hölder inequality and a Harnack inequality for parabolic inequality during the mean curvature flow in Riemannian manifolds. Consider the following differential inequality

$$(4.1) \quad \left(\frac{\partial}{\partial t} - \Delta \right) v \leq (f + C)v, \quad v \geq 0,$$

where the function f has bounded $L^q(M \times [0, T])$ -norm with $q > \frac{n+2}{2}$, and C is a fixed positive constant. Let $\eta(t, x)$ be a smooth function with the property that $\eta(0, x) = 0$ for all x .

Lemma 4.1. *Let*

$$(4.2) \quad C_0 \equiv C_0(q) = \|f\|_{L^q(M \times [0, T])}, \quad C_1 = \left(1 + \|H\|_{L^{\frac{2(n+3)}{3}}(M \times [0, T])}\right)^{\frac{n}{n+2}},$$

where $\beta > 1$ be a fixed number and $q > \frac{n+2}{2}$. Then, there exists a positive constant $C_a = C_a(n, q, C_0, C_1, C)$ such that

$$(4.3) \quad \begin{aligned} & \|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(M \times [0, T])} \\ & \leq C_a \Lambda(\beta)^{1+\nu} \left\| v^\beta \left(\eta^2 + |\nabla \eta|^2 + 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| \right) \right\|_{L^1(M \times [0, T])}, \end{aligned}$$

provided the function $(\eta v^{\frac{\beta}{2}})^{\frac{2(n-1)}{n-2}}$ satisfies conditions (3.2) and (3.3) for any $t \in [0, T)$, where

$$(4.4) \quad \nu = \frac{n+2}{2q - (n+2)},$$

and $\Lambda(\beta)$ is a positive constant depending on β such that $\Lambda(\beta) \geq 1$ if $\beta \geq 2$.

Remark 4.2. In fact, in Lemma 4.1 we can choose

$$(4.5) \quad C_a(n, q, C_0, C_1, C) = (2c_n C_0 C_1)^{1+\nu} + 2c_n C_1 (C+1).$$

Proof of Lemma 4.1. We mainly follow the ideas of the proof of Lemma 3.1 in [11]. First, we use $\eta^2 v^{\beta-1}$ as a test function in the following inequality

$$-\Delta v + \frac{\partial v}{\partial t} \leq (f + C)v.$$

Namely, for any $s \in (0, T]$,

$$(4.6) \quad \int_0^s \int_{M_t} (-\Delta v) \eta^2 v^{\beta-1} d\mu dt + \int_0^s \int_{M_t} \frac{\partial v}{\partial t} \eta^2 v^{\beta-1} d\mu dt \leq \int_0^s \int_{M_t} |f + C| \eta^2 v^\beta d\mu dt.$$

Note that, using the integration by parts we have

$$(4.7) \quad \int_{M_t} (-\Delta v) \eta^2 v^{\beta-1} d\mu = \int_{M_t} 2 \langle \nabla v, \nabla \eta \rangle \eta v^{\beta-1} d\mu + (\beta-1) \int_{M_t} \eta^2 v^{\beta-2} |\nabla v|^2 d\mu.$$

By the properties of η above, we get

$$(4.8) \quad \begin{aligned} \int_0^s \int_{M_t} \frac{\partial v}{\partial t} \eta^2 v^{\beta-1} d\mu dt &= \frac{1}{\beta} \int_0^s \int_{M_t} \frac{\partial(v^\beta)}{\partial t} \eta^2 d\mu dt \\ &= \frac{1}{\beta} \int_{M_t} v^\beta \eta^2 d\mu \Big|_0^s - \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta \partial_t (\eta^2 d\mu) dt \\ &= \frac{1}{\beta} \int_{M_s} v^\beta \eta^2 d\mu - \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta \left(2\eta \frac{\partial \eta}{\partial t} - H^2 \right) d\mu dt, \end{aligned}$$

where the last step we used the evolution of the volume form

$$(4.9) \quad \frac{\partial}{\partial t} d\mu = -H^2 d\mu.$$

Substituting (4.7) and (4.8) into (4.6) yields

$$\begin{aligned}
 (4.10) \quad & \int_0^s \int_{M_t} [2\langle \nabla v, \nabla \eta \rangle \eta v^{\beta-1} + (\beta-1)\eta^2 v^{\beta-2} |\nabla v|^2] d\mu dt + \frac{1}{\beta} \int_{M_s} v^\beta \eta^2 d\mu \\
 & \leq \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta 2\eta \frac{\partial \eta}{\partial t} d\mu dt + \int_0^s \int_{M_t} |f + C|\eta^2 v^\beta d\mu dt.
 \end{aligned}$$

In the following, we want to construct the quantity $(\frac{\partial}{\partial t} - \Delta)\eta$ to appear on the right hand side of (4.10). To achieve it, we find that integrating by parts yields

$$\begin{aligned}
 & \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta 2\eta \frac{\partial \eta}{\partial t} d\mu dt \\
 & = \frac{1}{\beta} \int_0^s \int_{M_t} \left[v^\beta 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta + v^\beta 2\eta \Delta \eta \right] d\mu dt \\
 & = \frac{1}{\beta} \int_0^s \int_{M_t} \left[v^\beta 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta - 2\nabla(v^\beta \eta) \nabla \eta \right] d\mu dt \\
 & = \frac{1}{\beta} \int_0^s \int_{M_t} \left[v^\beta 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta - 2v^\beta |\nabla \eta|^2 - 2\beta \langle \nabla v, \nabla \eta \rangle \eta v^{\beta-1} \right] d\mu dt \\
 & \leq \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta d\mu dt - \int_0^s \int_{M_t} 2\eta \langle \nabla v, \nabla \eta \rangle v^{\beta-1} d\mu dt.
 \end{aligned}$$

Hence (4.10) becomes

$$\begin{aligned}
 (4.11) \quad & \int_0^s \int_{M_t} [4\langle \nabla v, \nabla \eta \rangle \eta v^{\beta-1} + (\beta-1)\eta^2 v^{\beta-2} |\nabla v|^2] d\mu dt + \frac{1}{\beta} \int_{M_s} v^\beta \eta^2 d\mu \\
 & \leq \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| d\mu dt + \int_0^s \int_{M_t} |f + C|\eta^2 v^\beta d\mu dt.
 \end{aligned}$$

Note that the Cauchy-Schwartz inequality implies

$$\begin{aligned}
 \int_0^s \int_{M_t} 4\langle \nabla v, \nabla \eta \rangle \eta v^{\beta-1} d\mu dt & \geq -2\epsilon^2 \int_0^s \int_{M_t} \eta^2 v^{\beta-2} |\nabla v|^2 d\mu dt \\
 & \quad - \frac{2}{\epsilon^2} \int_0^s \int_{M_t} v^\beta |\nabla \eta|^2 d\mu dt.
 \end{aligned}$$

Hence (4.11) becomes

$$\begin{aligned}
 & \int_0^s \int_{M_t} (\beta-1-2\epsilon^2)\eta^2 v^{\beta-2} |\nabla v|^2 d\mu dt + \frac{1}{\beta} \int_{M_s} v^\beta \eta^2 d\mu \\
 & \leq \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| d\mu dt + \int_0^s \int_{M_t} |f + C|\eta^2 v^\beta d\mu dt \\
 & \quad + \frac{2}{\epsilon^2} \int_0^s \int_{M_t} v^\beta |\nabla \eta|^2 d\mu dt.
 \end{aligned}$$

Noticing the relation

$$|\nabla(v^{\beta/2})|^2 = \frac{\beta^2}{4} v^{\beta-2} |\nabla v|^2,$$

if we choose $\epsilon^2 = \frac{\beta-1}{4}$, then

$$\begin{aligned} & 2 \left(1 - \frac{1}{\beta}\right) \int_0^s \int_{M_t} \eta^2 |\nabla(v^{\beta/2})|^2 d\mu dt + \int_{M_s} v^\beta \eta^2 d\mu \\ & \leq \int_0^s \int_{M_t} v^\beta 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| d\mu dt + \beta \int_0^s \int_{M_t} |f + C| \eta^2 v^\beta d\mu dt \\ & \quad + \frac{8\beta}{\beta-1} \int_0^s \int_{M_t} v^\beta |\nabla \eta|^2 d\mu dt. \end{aligned}$$

Combining the above estimate with

$$|\nabla(\eta v^{\beta/2})|^2 \leq 2\eta^2 |\nabla(v^{\beta/2})|^2 + 2v^\beta |\nabla \eta|^2$$

implies

$$\begin{aligned} & \left(1 - \frac{1}{\beta}\right) \int_0^s \int_{M_t} |\nabla(\eta v^{\beta/2})|^2 d\mu dt + \int_{M_s} v^\beta \eta^2 d\mu \\ & \leq \int_0^s \int_{M_t} v^\beta 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| d\mu dt + \beta \int_0^s \int_{M_t} |f + C| \eta^2 v^\beta d\mu dt \\ & \quad + 8 \left(\frac{\beta}{\beta-1} + \frac{\beta-1}{\beta} \right) \int_0^s \int_{M_t} v^\beta |\nabla \eta|^2 d\mu dt. \end{aligned}$$

It follows that, for some $\Lambda(\beta) \geq 1$ (for example, we can choose $\Lambda(\beta) = 100\beta$ if $\beta \geq 2$),

$$\begin{aligned} & \int_0^s \int_{M_t} |\nabla(\eta v^{\beta/2})|^2 d\mu dt + \int_{M_s} v^\beta \eta^2 d\mu \\ & \leq \Lambda(\beta) \int_0^s \int_{M_t} v^\beta \left\{ 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| + |\nabla \eta|^2 \right\} d\mu dt \\ & \quad + \Lambda(\beta) \int_0^s \int_{M_t} |f + C| \eta^2 v^\beta d\mu dt \\ & \leq \Lambda(\beta) \int_0^s \int_{M_t} v^\beta \left\{ 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| + |\nabla \eta|^2 \right\} d\mu dt \\ & \quad + \Lambda(\beta) \left(\|f\|_{L^q(M \times [0, T])} \|\eta^2 v^\beta\|_{L^{\frac{q}{q-1}}(M \times [0, T])} + \int_0^s \int_{M_t} C \eta^2 v^\beta d\mu dt \right) \\ & = \Lambda(\beta) \int_0^s \int_{M_t} v^\beta \left\{ 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| + |\nabla \eta|^2 \right\} d\mu dt \\ & \quad + \Lambda(\beta) \left(C_0 \|\eta^2 v^\beta\|_{L^{\frac{q}{q-1}}(M \times [0, T])} + C \|\eta^2 v^\beta\|_{L^1(M \times [0, T])} \right) \\ & =: B. \end{aligned}$$

From the above inequalities, we conclude

$$(4.12) \quad \max_{0 \leq s \leq T} \int_{M_s} \eta^2 v^\beta d\mu \leq B$$

and

$$(4.13) \quad \int_0^T \int_{M_t} |\nabla(\eta v^{\beta/2})|^2 d\mu dt \leq B.$$

If the function $(\eta v^{\frac{\beta}{2}})^{\frac{2(n-1)}{n-2}}$ satisfies conditions (3.2) and (3.3) for any $t \in [0, T)$, applying Proposition 3.5 to $\eta v^{\beta/2}$, we have the following estimates

$$\begin{aligned}
& \|\eta^2 v^\beta\|_{L^{(n+2)/n}(M \times [0, T))}^{(n+2)/n} \\
&= \|\eta v^{\beta/2}\|_{L^{2(n+2)/n}(M \times [0, T))}^{2(n+2)/n} \\
&\leq c_n \max_{0 \leq t \leq T} \|\eta v^{\beta/2}\|_{L^2(M_t)}^{4/n} \\
&\quad \times \left(\|\nabla(\eta v^{\beta/2})\|_{L^2(M \times [0, T))}^2 + \max_{0 \leq t \leq T} \|\eta v^{\beta/2}\|_{L^2(M_t)}^2 \|H\|_{L^{n+2}(M \times [0, T))}^{n+2} \right) \\
&\leq c_n B^{2/n} \left(B + B \|H\|_{L^{n+2}(M \times [0, T))}^{n+2} \right) \\
&= c_n B^{\frac{n+2}{n}} \left(1 + \|H\|_{L^{n+2}(M \times [0, T))}^{n+2} \right).
\end{aligned}$$

For the convenient of writing, we let $S := M \times [0, T)$ and let the norm $\|\cdot\|_{L^p(M \times [0, T))}$ be shortly denoted by $\|\cdot\|_{L^p(S)}$. Then by the definition of B , the previous estimate can be rewritten as

$$\begin{aligned}
(4.14) \quad & \|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} \leq c_n B \left(1 + \|H\|_{L^{n+2}(S)}^{n+2} \right)^{\frac{n}{n+2}} \\
&= c_n C_1 \Lambda(\beta) \int_0^T \int_{M_t} v^\beta \left\{ 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| + |\nabla \eta|^2 \right\} d\mu dt \\
&\quad + c_n C_1 \Lambda(\beta) \left(C_0 \|\eta^2 v^\beta\|_{L^{\frac{q}{q-1}}(S)} + C \|\eta^2 v^\beta\|_{L^1(S)} \right).
\end{aligned}$$

Since $1 < \frac{q}{q-1} < \frac{n+2}{n}$, we have the interpolation inequality

$$\|\eta^2 v^\beta\|_{L^{\frac{q}{q-1}}(S)} \leq \epsilon \|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} + \epsilon^{-\nu} \|\eta^2 v^\beta\|_{L^1(S)},$$

where $\nu = \frac{n+2}{2q-(n+2)}$. Then plugging this inequality into (4.14) yields

$$\begin{aligned}
& [1 - c_n \Lambda(\beta) C_0 C_1 \epsilon] \|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} \\
&\leq c_n C_1 \Lambda(\beta) \left[(C_0 \epsilon^{-\nu} + C) \|\eta^2 v^\beta\|_{L^1(S)} + \left\| v^\beta \left(|\nabla \eta|^2 + 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right) \right\|_{L^1(S)} \right].
\end{aligned}$$

If we further choose $\epsilon = \frac{1}{2\Lambda(\beta)c_n C_0 C_1}$, then

$$\begin{aligned}
& \|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} \\
&\leq 2c_n C_1 \Lambda(\beta) [C_0 (2\Lambda(\beta)c_n C_0 C_1)^\nu + C] \|\eta^2 v^\beta\|_{L^1(S)} \\
&\quad + 2c_n C_1 \Lambda(\beta) \left\| v^\beta \left(|\nabla \eta|^2 + 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right) \right\|_{L^1(S)} \\
&\leq C_a(n, q, C_0, C_1, C) \Lambda(\beta)^{1+\nu} \left\| v^\beta \left(\eta^2 + |\nabla \eta|^2 + 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right) \right\|_{L^1(S)},
\end{aligned}$$

where

$$C_a(n, q, C_0, C_1, C) = (2c_n C_0 C_1)^{1+\nu} + 2c_n C_1 (C + 1).$$

Therefore we obtain a soft reverse Hölder inequality

$$\begin{aligned} & \|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} \\ & \leq C_a(n, q, C_0, C_1, C) \Lambda(\beta)^{1+\nu} \left\| v^\beta \left(\eta^2 + |\nabla \eta|^2 + 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| \right) \right\|_{L^1(S)} \end{aligned}$$

provided the function $(\eta v^{\frac{\beta}{2}})^{\frac{2(n-1)}{n-2}}$ satisfies conditions (3.2) and (3.3) for any $t \in [0, T)$. \square

Next, we shall show that an L^∞ -norm of v over a smaller set can be bounded by an L^β -norm of v on a bigger set, where $\beta \geq 2$. Fix $x_0 \in N^{n+1}$. Consider the following space-time sets:

$$D = \cup_{0 \leq t \leq 1} (B(x_0, 1) \cap M_t); \quad D' = \cup_{\frac{1}{12} \leq t \leq 1} (B(x_0, \frac{1}{2^p}) \cap M_t),$$

where $p \geq 1$ is large enough, depending on n , $\max_{x \in M_0} |A|$ and N , to be described later. Then, we have the following Harnack inequality.

Lemma 4.3. *Consider the equation (4.1) with $T = 1$. Let us denote by $\lambda = \frac{n+2}{n}$, let $q > \frac{n+2}{2}$ and $\beta \geq 2$. Then, there exists a constant $C_b = C_b(n, q, \beta, C_0, C_1, C)$ such that*

$$(4.15) \quad \|v\|_{L^\infty(D')} \leq C_b \|v\|_{L^\beta(D)}.$$

In the above inequalities, C_0 and C_1 are defined by (4.2).

Remark 4.4. In fact, in Lemma 4.3 we can choose

$$(4.16) \quad C_b(n, p, q, \beta, C_0, C_1, C) = (4^p \lambda^{1+\nu} C_z \beta^{1+\nu})^{\frac{n^2}{\beta}},$$

where

$$(4.17) \quad C_z(n, q, C_0, C_1, C) := 4^2 \times 100^{1+\nu} c_n C_a(n, q, C_0, C_1, C).$$

Proof of Lemma 4.3. The proof of this lemma, using Lemma 4.1 and Moser iteration, is similar to that of Lemma 5.2 in [12]. If we set

$$D_k = \cup_{t_k \leq t \leq 1} (B(x_0, r_k) \cap M_t),$$

where

$$r_k = \frac{1}{2^p} + \frac{1}{2^{p+k+1}}; \quad t_k = \frac{1}{12} \left(1 - \frac{1}{4^{p+k}} \right),$$

then

$$\rho_k := r_{k-1} - r_k = \frac{1}{2^{p+k+1}}; \quad t_k - t_{k-1} = \rho_k^2.$$

Following Ecker [4], we choose a test function $\eta_k = \eta_k(t, x)$ of the form

$$(4.18) \quad \eta_k(t, x) = \varphi_{\rho_k}(t) \times \psi_{\rho_k}(|x - x_0|^2),$$

where the function φ_{ρ_k} satisfies

$$\varphi_{\rho_k}(t) = \begin{cases} 1 & \text{if } t_k \leq t \leq 1, \\ \in [0, 1] & \text{if } t_{k-1} \leq t \leq t_k, \\ 0 & \text{if } t \leq t_{k-1}. \end{cases}$$

and

$$|\varphi'_{\rho_k}|(t) \leq \frac{c_n}{\rho_k^2};$$

whereas the function $\psi_{\rho_k}(s)$ satisfies

$$\psi_{\rho_k}(s) = \begin{cases} 0 & \text{if } s \geq r_{k-1}^2, \\ \in [0, 1] & \text{if } r_k^2 \leq s \leq r_{k-1}^2, \\ 1 & \text{if } s \leq r_k^2. \end{cases}$$

and

$$|\psi'_{\rho_k}|(s) \leq \frac{c_n}{\rho_k^2}.$$

By the definition, we can easily check that

$$0 \leq \eta_k \leq 1; \quad \eta_k \equiv 1 \text{ in } D_k; \quad \eta_k \equiv 0 \text{ outside } D_{k-1}$$

and

$$(4.19) \quad \sup_{t \in [0, 1]} \sup_{x \in M_t} \left(\eta_k^2 + |\nabla \eta_k|^2 + 2\eta_k \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta_k \right| \right) \leq \frac{c_n}{\rho_k^2} = c_n 4^{p+k+1},$$

where $\eta_k = \eta_k(t, x)$. Here we claim that if $\beta \geq 2$, let $\Lambda(\beta) = 100\beta$. Now using Lemma 4.1, we have

$$(4.20) \quad \begin{aligned} \|v^\beta\|_{L^{\frac{n+2}{n}}(D_k)} &\leq \|\eta_k^2 v^\beta\|_{L^{\frac{n+2}{n}}(M \times [0, T])} \\ &\leq C_a \Lambda(\beta)^{1+\nu} \int_0^T \int_{M_t} v^\beta \left(\eta_k^2 + |\nabla \eta_k|^2 + 2\eta_k \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta_k \right| \right) d\mu dt \\ &\leq c_n 4^{p+k+1} C_a \Lambda(\beta)^{1+\nu} \int_{D_{k-1}} v^\beta d\mu dt \\ &:= C_z(n, q, C_0, C_1, C) 4^{p+k-1} \beta^{1+\nu} \|v^\beta\|_{L^1(D_{k-1})}, \end{aligned}$$

as long as p is sufficient large so that the function $(\eta_k v^{\frac{\beta}{2}})^{\frac{2(n-1)}{n-2}}$ naturally satisfies conditions (3.2) and (3.3) for any $t \in [0, T]$, where the third line results from the inequality (4.19) and the definition of ρ_k . Note that we can choose

$$\Lambda(\beta) = 100\beta, \quad C_z(n, q, C_0, C_1, C) = 4^2 \times 100^{1+\nu} c_n C_a.$$

Now we explain why $(\eta_k v^{\frac{\beta}{2}})^{\frac{2(n-1)}{n-2}}$ naturally satisfies conditions (3.2) and (3.3) for any $t \in [0, T]$ when p is large enough. In fact under mean curvature flow, we observe that

$$\text{Vol}_{g(t)}(B(R)) \leq \text{Vol}_{g(0)}(B(R))$$

for any $t \in [0, T]$ by (4.9). For $g(0)$, there is a non-positive constant K depending on n , $\max_{x \in M_0} |A|$ and N such that the sectional curvature of M_0 is bounded from below by K . Then the Bishop-Gromov volume comparison theorem implies

$$\text{Vol}_{g(0)}(B(R)) \leq \text{Vol}_K(B(R)),$$

where $\text{Vol}_K(B(R))$ denotes the volume of the ball with radius R in the n -dimensional complete simply connected space form with constant curvature K . Hence

$$\text{Vol}_{g(t)}(B(R)) \leq \text{Vol}_K(B(R)).$$

Therefore, we can choose R sufficient small such that

$$b^2(1 - \alpha)^{-2/n} (\omega_n^{-1} \text{Vol}_K(B(R)))^{2/n} \leq 1$$

and

$$2\rho_0 \leq \bar{R}(M),$$

where ρ_0 is defined by (3.4). Here the sufficient small R can be achieved by choosing a sufficient large p . Hence this explanation shows that the function $(\eta_k v^{\frac{\beta}{2}})^{\frac{2(n-1)}{n-2}}$ naturally satisfies conditions (3.2) and (3.3) for any $t \in [0, T)$ as long as p is large enough.

For simplicity, let us denote by $C_z = C_z(n, q, C_0, C_1, C)$. Then (4.20) can be simplified as follows:

$$(4.21) \quad \|v\|_{L^{\frac{n+2}{n}\beta}(D_k)} \leq C_z^{\frac{1}{\beta}} 4^{\frac{p+k-1}{\beta}} \beta^{\frac{1+\nu}{\beta}} \|v\|_{L^\beta(D_{k-1})}.$$

This inequality is the key estimate for our Moser iteration process. If we set $\lambda = \frac{n+2}{n}$ and replace β by $\lambda^{k-1}\beta$ in (4.21), then

$$\begin{aligned} \|v\|_{L^{\beta\lambda^k}(D_k)} &\leq C_z^{\frac{1}{\beta\lambda^{k-1}}} 4^{\frac{p+k-1}{\beta\lambda^{k-1}}} (\lambda^{k-1}\beta)^{\frac{1+\nu}{\beta\lambda^{k-1}}} \|v\|_{L^{\beta\lambda^{k-1}}(D_{k-1})} \\ &\leq C_z^{\frac{1}{\beta\lambda^{k-1}}} (4^p)^{\frac{k-1}{\beta\lambda^{k-1}}} (\lambda^{k-1}\beta)^{\frac{1+\nu}{\beta\lambda^{k-1}}} \|v\|_{L^{\beta\lambda^{k-1}}(D_{k-1})}. \end{aligned}$$

It follows by iteration that for all $k_0 \geq 0$

$$(4.22) \quad \begin{aligned} \|v\|_{L^{\beta\lambda^k}(D_k)} &\leq \left(C_z^{\frac{1}{\beta}} \beta^{\frac{1+\nu}{\beta}} \right)^{\sum_{j=k_0}^{k-1} \frac{1}{\lambda^j}} \left(4^{\frac{p}{\beta}} \cdot \lambda^{\frac{1+\nu}{\beta}} \right)^{\sum_{j=k_0}^{k-1} \frac{j}{\lambda^j}} \|v\|_{L^{\beta\lambda^{k_0}}(D_{k_0})} \\ &\leq C_b(n, p, q, \beta, C_0, C_1, C) \|v\|_{L^{\beta\lambda^{k_0}}(D_{k_0})}, \end{aligned}$$

where we can choose

$$C_b(n, p, q, \beta, C_0, C_1, C) = (4^p \lambda^{1+\nu} C_z \beta^{1+\nu})^{\frac{n^2}{\beta}},$$

since

$$\sum_{j=0}^{\infty} \frac{j}{\lambda^j} = \frac{\lambda}{(\lambda-1)^2} = \frac{n(n+2)}{4} \leq n^2.$$

Note that $D_0 \subseteq D$ and $D' \subset D_k$ for all positive integer k . So inequality (4.22) yields

$$\begin{aligned} \|v\|_{L^{\beta\lambda^k}(D')} &\leq C_b(n, p, q, \beta, C_0, C_1, C) \|v\|_{L^\beta(D_0)} \\ &\leq C_b(n, p, q, \beta, C_0, C_1, C) \|v\|_{L^\beta(D)}. \end{aligned}$$

Letting $k \rightarrow \infty$, we get

$$\|v\|_{L^\infty(D')} \leq C_b(n, p, q, \beta, C_0, C_1, C) \|v\|_{L^\beta(D)}.$$

□

5. BOUNDING THE SECOND FUNDAMENTAL FORM

In this section, we will prove that the second fundamental form $A(\cdot, t)$ can be bounded by its certain integral sense. First we establish a rescaled version.

Proposition 5.1. *Let $F_t : M^n \rightarrow N^{n+1}$ be a solution to the mean curvature flow (1.1) of compact hypersurfaces of a locally symmetric Riemannian manifold N^{n+1} with bounded geometry. If there exists a universal constant c_0 such that*

$$(5.1) \quad \int_0^1 \int_{M_t} |A|^{n+3} d\mu dt \leq c_0,$$

then

$$(5.2) \quad \sup_{\frac{1}{2} \leq t \leq 1} \sup_{x \in M_t} |A(x, t)| \leq 1.$$

Proof of Proposition 5.1. We follow the lines of the proof of Lemma 4.1 in [11]. In Section 2, we have the following evolution equation:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) |A|^2 &= -2|\nabla A|^2 + 2|A|^2 (|A|^2 + \bar{Ric}(\nu, \nu)) \\ &\quad - 4(h^{ij}h_j^m \bar{R}_{mli}{}^l - h^{ij}h^{lm} \bar{R}_{milj}) \\ &\quad - 2h^{ij}(\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l). \end{aligned}$$

Since N is locally symmetric by assumption, we have $\bar{\nabla} \bar{R}m = 0$. So the above equation becomes

$$(5.3) \quad \begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) |A|^2 &= -2|\nabla A|^2 + 2|A|^2 (|A|^2 + \bar{Ric}(\nu, \nu)) \\ &\quad - 4(h^{ij}h_j^m \bar{R}_{mli}{}^l - h^{ij}h^{lm} \bar{R}_{milj}) \end{aligned}$$

Furthermore, using the bounds on the geometry of N we have

$$\left(\frac{\partial}{\partial t} - \Delta \right) |A|^2 \leq 2|A|^4 + C|A|^2,$$

where C is a positive constant, depending on N . If we set $v = |A|^2$ and $f = 2|A|^2$, then

$$\left(\frac{\partial}{\partial t} - \Delta \right) v \leq (f + C)v.$$

Now we will use Lemma 4.3 to the above equation. Here we let $q = \frac{n+3}{2}$ and $\beta = \frac{n+3}{2}$. (4.4) and (4.5) imply

$$C_a = (2c_n C_0 C_1)^{n+3} + 2c_n C_1 (C + 1).$$

Meanwhile from (4.16) and (4.17), we have

$$C_b = c_n(p) C_z^{\frac{2n^2}{n+3}} = c_n(p) C_a^{\frac{2n^2}{n+3}} = c_n(p) [(C_0 C_1)^{n+3} + C_1 (C + 1)]^{\frac{2n^2}{n+3}},$$

where $c_n(p)$ only depends on n and p . We also know that

$$(5.4) \quad \begin{aligned} C_0 &= 2 \left[\int_0^1 \int_{M_t} |A|^{2q} d\mu dt \right]^{1/q} \\ &\leq 2c_0^{\frac{2}{n+3}} \end{aligned}$$

and

$$(5.5) \quad \begin{aligned} C_1 &= \left[1 + \left(\int_0^1 \int_{M_t} |H|^{n+3} d\mu dt \right)^{\frac{2}{3}} \right]^{\frac{n}{n+2}} \\ &\leq \left[1 + \left(\int_0^1 \int_{M_t} n^{\frac{n+3}{2}} |A|^{n+3} d\mu dt \right)^{\frac{2}{3}} \right]^{\frac{n}{n+2}} \\ &\leq \left(1 + n^{\frac{n+3}{3}} c_0^{\frac{2}{3}} \right)^{\frac{n}{n+2}}, \end{aligned}$$

where we used inequality $|H|^2 \leq n|A|^2$. Hence by Lemma 4.3, and using (5.4) and (5.5), we have

$$\begin{aligned}
 \|v\|_{L^\infty(D')} &\leq C_b \|v\|_{L^\beta(D)} \\
 &= c_n(p) [(C_0 C_1)^{n+3} + C_1(C+1)]^{\frac{2n^2}{n+3}} \|v\|_{L^\beta(D)} \\
 &\leq c_n(p) [(C_0 C_1)^{n+3} + C_1(C+1)]^{\frac{2n^2}{n+3}} c_0^{\frac{2}{n+3}} \\
 &\leq 1
 \end{aligned}
 \tag{5.6}$$

if c_0 is sufficient small. Note that $c_n(p)$ does not depend on c_0 . \square

Using Proposition 5.1, and following the proof of Proposition 1.1 in [11], we have the following key result for proving Theorem 1.6.

Proposition 5.2. *Let $F_t : M^n \rightarrow N^{n+1}$ be a solution to the mean curvature flow (1.1) of compact hypersurfaces of a locally symmetric Riemannian manifold N^{n+1} with bounded geometry. For all $\lambda \in (0, 1]$, there exists a constant c_λ such that for all $T \geq \lambda$, we have*

$$\sup_{x \in M_T} |A(x, T)| \leq c_\lambda \left(1 + \int_0^T \int_{M_t} |A|^{n+3} d\mu dt \right).$$

Proof of Theorem 5.2. The proof is the same as that of Proposition 1.1 in [11]. \square

6. PROOF OF THEOREM 1.6

In this section, we will follow the Le's method in [11] and prove Theorem 1.6 in introduction. The proof is the same as the Le's proof. For the completeness, we still give the details of the proof here.

Proof of Theorem 1.6. Fix $\tau_1 < T$ such that $0 < \tau_1 < 1$. By Proposition 5.2, for any $t \geq \tau_1$, there is a universal constant c depending on τ_1 , such that

$$(6.1) \quad \sup_{x \in M_t} |A(x, t)| \leq c(\tau_1) \left(1 + \int_0^t \int_{M_s} |A|^{n+3} d\mu ds \right).$$

We set $f(t) = \sup_{x \in M_t} |A(x, t)|$, $\Psi(s) = s \log(a + s)$ and

$$G(s) = \int_{M_s} \frac{|A|^{n+2}}{\log(a + |A|)} d\mu,$$

where a is a fixed constant and $a \geq 1$.

Note that Ψ is an increasing function. Then (6.1) can be read as

$$\begin{aligned}
 f(t) &\leq c(\tau_1) \left(1 + \int_0^t \int_{M_s} \Psi(|A|) \frac{|A|^{n+2}}{\log(a + |A|)} d\mu ds \right) \\
 &\leq c(\tau_1) \left[1 + \int_0^t \Psi \left(\sup_{x \in M_s} |A(x, s)| \right) \int_{M_s} \frac{|A|^{n+2}}{\log(a + |A|)} d\mu ds \right] \\
 &= c(\tau_1) \left(1 + \int_0^t \Psi(f(s)) G(s) ds \right).
 \end{aligned}$$

If we set

$$h(t) = c(\tau_1) \left(1 + \int_0^t \Psi(f(s)) G(s) ds \right),$$

then for $t \geq \tau_1$

$$f(t) \leq h(t)$$

and

$$h'(t) = c(\tau_1)\Psi(f(t))G(t) \leq c(\tau_1)\Psi(h(t))G(t).$$

If we further set

$$\tilde{\Psi}(y) = \int_c^y \frac{1}{\Psi(s)} ds,$$

then for $t \geq \tau_1$,

$$\tilde{\Psi}(h(t)) - \tilde{\Psi}(h(\tau_1)) \leq c(\tau_1) \int_{\tau_1}^t G(s) ds \leq c(\tau_1) \int_0^T G(s) ds < \infty.$$

Note that $h(\tau_1)$ is finite. Therefore

$$(6.2) \quad \sup_{\tau_1 \leq t < T} \tilde{\Psi}(h(t)) \leq \tilde{\Psi}(h(\tau_1)) + c(\tau_1) \int_0^T G(s) ds < \infty.$$

Since $\int_c^\infty \frac{ds}{\Psi(s)} = \int_c^\infty \frac{ds}{s \log(a+s)} = \infty$, by (6.2) we deduce that

$$\sup_{\tau_1 \leq t < T} h(t) < \infty.$$

Hence

$$\sup_{\tau_1 \leq t < T} f(t) < \infty.$$

Namely, $\sup_{x \in M_t} |A(x, t)| < \infty$ for $0 \leq t < T$. By Theorem 1.4 in introduction, the mean curvature flow can be extended past T . \square

Just as Le's said in [11], from the proof of Theorem 1.6, we have another subcritical quantities concerning the second fundamental and obtain a similar extension result. For example: We choose $\Psi(s) = s \log(a + s^b)$ and

$$G(s) = \int_{M_s} \frac{|A|^{n+2}}{\log(a + |A|^b)} d\mu,$$

where b is a fixed constant and $b \geq 1$. Notice that $\int_c^\infty \frac{ds}{s \log(a + s^b)} = \infty$. Following the proof of Theorem 1.6, we have

Theorem 6.1. *Suppose $T < \infty$ is the first singularity time for a mean curvature flow (1.1) of compact hypersurfaces of a locally symmetric Riemannian manifold N^{n+1} with bounded geometry. If*

$$\int_0^T \int_{M_t} \frac{|A|^{n+2}}{\log(a + |A|^b)} d\mu dt < \infty,$$

where a and b are fixed constants and $a, b \geq 1$, then this flow can be extended past time T .

7. PROOF OF COROLLARY 1.7

In this section, we shall apply Theorem 1.6 to give the proof of Corollary 1.7 stated in the introduction.

Proof of Corollary 1.7. By Hölder's inequality, we know that $\|A\|_{L^\alpha(M \times [0, T])} < \infty$, where $\alpha > n+2$, implies $\|A\|_{L^{n+2}(M \times [0, T])} < \infty$. So we only need to prove Corollary 1.7 for $\alpha = n+2$.

From Theorem 1.6, we see that if $T < \infty$ is the first singularity time for a mean curvature flow (1.1) of a compact hypersurface of a locally symmetric Riemannian manifold N^{n+1} with bounded geometry, and if

$$\int_0^T \int_{M_t} \frac{|A|^{n+2}}{\log(a + |A|)} d\mu dt < \infty$$

where a is a fixed constant and $a \geq 1$, then the mean curvature flow can be extended past T . Now if we choose $a = 100$, then

$$\frac{1}{\log(100 + |A|)} \leq \frac{1}{2}.$$

Namely,

$$\int_0^T \int_{M_t} \frac{|A|^{n+2}}{\log(100 + |A|)} d\mu dt \leq \frac{1}{2} \int_0^T \int_{M_t} |A|^{n+2} d\mu dt.$$

Hence the assumption of Corollary 1.7 guarantees that the mean curvature flow can be extended past T . \square

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